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Pole Solution in Six Dimensions and Mass Hierarchy ¹

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Abstract

A solution of the 6D gravitational model, which has the pole configuration, is found. The vacuum setting is done by the 6D Higgs potential and the solution is for a family of the vacuum and boundary parameters. The boundary condition is solved by the $1/k^2$ -expansion (thin pole expansion) where $1/k$ is the *thickness* of the pole. The obtained analytic solution is checked by the numerical method. This is a dimensional reduction model from 6D to 4D by use of the soliton solution (brane world). It is regarded as a higher dimensional version of the Randall-Sundrum 5D model. The mass hierarchy is examined. Especially the *geometrical see-saw* mass relation is obtained. Some physical quantities in 4D world such as the Planck mass, the cosmological constant, and fermion masses are focussed. Comparison with the 5D model is made.

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1 Introduction

The higher dimensional approach is a natural way to analyze the 4D physics in the geometrical standpoint. The history traces back to Kaluza-Klein in 1921[1, 2] where the electro-magnetic force is amalgamated into the gravitational force in the 5D theory. Stimulated by the recent development of the string and D-brane theories, a new type compactification mechanism was invented by Randall and Sundrum[3, 4]. The domain wall configuration in 5D space-time, which is a kink solution in the extra dimension, is exploited. Before Randall-Sundrum(RS), the supersymmetry is the popular way to extend the standard model. The D-brane inspired model has provided us with new possibilities for the extension. It has some advantages in the hierarchy problem and the chiral problem.

A moderate view to the string or D-brane theory is to regard the formalism as a *regularization* of the field theory. In this standpoint, the extendedness, in the space-time, of an object is the essence of the *singular-free* behavior. The non-commutative geometry, which is a recent trend in the field theory, can be regarded as a systematic way to introduce the extendedness. In the ordinary (commutative) geometry, the soliton configuration can do the same role. We present a 6D soliton solution, and show that it provides a new dimensional reduction mechanism.

In the research of the domain world physics, the main theoretical focuses, at present, are 1) Consistency, 2) Stability and 3) Localization. The first one is the basic thing in all theoretical analysis. In the analysis of the domain world physics, the delta-function type special configuration (singular space) is often assumed without properly solving the field equation with some boundary condition. This is a main origin of the necessity of the consistency check. In some limited situation, it can be accepted and is indeed useful in that the whole physical image can be easily introduced. The approach is, however, not admitted as the right configuration in the general case. It is sometimes discussed that some no-go theorem exists in some models[5, 6, 7]. Especially the rigorous treatment is required in the analysis related to the cosmological constant and the boundary condition[8, 9, 10]. Recently in order to check such consistency for a wide-range brane-world models, some useful sum rule is presented[11]. The present approach treats the configuration in the general setting and the model is based on the field theory. Therefore it basically passes the consistency check. Next item is the stability problem. The solution should satisfy the stability condition. It is generally rather difficult to analyze because the solution should be examined

from some general configuration depending on parameters such as the distance between the branes. One interesting idea is proposed by Goldberger and Wise[12], where the radion field is utilized. In the present approach, the stability is guaranteed by the *boundary condition*. This is the popular situation in the soliton physics. The third focus is whether the ordinary fields (matter fields, gauge fields, graviton, etc.) are localized in some narrow region in the extra space. This is necessary to be consistent with the fact that we have so far not observed the extra dimensions (requirement for the dimensional reduction). In the present analysis, no real matter fields are taken into account. Only 6D Higgs scalar is introduced to define the (classical) vacuum clearly. The localization of other fields, in the present analysis, is assumed valid based on other references[13, 14] and on the analogy to the lattice fermion situation.

We have some reasons to investigate the higher dimensional generalization of the RS-model. Firstly some references suggest the difficulty of making a realistic stable 5D domain world[5, 6]. One of main causes looks its odd dimensionality. Secondly, in the solitonic approach, various extended objects (wall, string, monopole, ...) appear[15]. It is a natural direction to investigate the possibility of other type configuration than wall. Thirdly in the work by Callan and Harvey[16], they interpret the anomaly phenomena in 4D world in the higher dimensions. They exploit the physics of fermion zero modes on strings in $2n+2$ dimensions and domain walls in $2n+1$ dimensions. The result is consistent with the anomaly relations between different dimensions by Alvarez Gaumé and Witten[17]. 6D model is necessary, in addition to 5D, to understand the whole anomaly structure. We consider the brane world model has great advantages about the anomaly and chiral problems. Finally, in the present rapid progress of the noncommutative geometry, the extra space has even dimensions. We suppose the present approach has something common with such approach.

The 6D models of the hierarchy problem have already been discussed in various ways[18, 19, 20, 21]. Main different points from those works are 1) the present approach puts emphasis on the *thickness* of the configuration, 2) the 6D(bulk) Higgs fields are introduced to clearly define the (classical) vacua, which are necessary for the soliton boundary conditions, 3) we treat the warp factors and the scalar (Higgs) fields on the equal footing, 4) the boundary condition is systematically solved and 5) supersymmetry is not taken into account.

In Sec.2 we introduce the 6D model and derive a solution of the 6D classical Einstein equation in Sec.3. It is a one parameter family of soliton

solution with the pole configuration. The asymptotic forms, in the dimensional reduction, of the vacuum parameters are obtained in Sec.4. In Sec.5, some physical quantities, such as the 4D Planck mass, the 4D cosmological constant and fermion masses, are explained. The *see-saw* relation is obtained there. Some order estimation is also done. The precise form of the solution is obtained by analytically solving the boundary condition in Sec.6. It is demonstrated that the vacuum parameters are fixed to be some numbers by one input (boundary) condition. The obtained solution is further confirmed by the numerical method in Sec.7. Finally we conclude in Sec.8. In app.A we explain, in detail, the derivation of the solution treated in Sec.3. Some detail about the numerical analysis of Sec.7 is explained in App.B.

2 Model set-up

We start with the 6D gravitational theory, where the metric is Lorentzian, with the 6D Higgs potential.

$$S[G_{AB}, \Phi] = \int d^6 X \sqrt{-G} \left(-\frac{1}{2} M^4 \hat{R} - G^{AB} \partial_A \Phi^* \partial_B \Phi - V(\Phi^*, \Phi) \right) ,$$

$$V(\Phi^*, \Phi) = \frac{\lambda}{4} (|\Phi|^2 - v_0^2)^2 + \Lambda , \quad (1)$$

where X^A ($A = 0, 1, 2, 3, 4, 5$) is the 6D coordinates and we also use the notation $(X^A) \equiv (x^\mu, \rho, \varphi)$, $\mu = 0, 1, 2, 3$. x^μ 's are regarded as our world coordinates, whereas $(X^4, X^5) = (\rho, \varphi)$ the extra ones. The extra (spacial) coordinates are taken to be polar ones: $0 \leq \rho < \infty$, $0 \leq \varphi < 2\pi$. ρ will be regarded as a "radius" and φ as an "angle" around the 4D pole-world. See Fig.1. The signature of the 6D metric G_{AB} is $(- + + + +)$. Φ is the 6D *complex* scalar field, $G = \det G_{AB}$, \hat{R} is the 6D Riemannian scalar curvature. $M(> 0)$ is the 6D Planck mass and is regarded as the *fundamental scale* of this dimensional reduction scenario. $V(\Phi^*, \Phi)$ is the Higgs potential and defines the (classical) vacuum in the 6D world. The three parameters λ, v_0 and Λ in the potential V are called here *vacuum parameters*. $\lambda(> 0)$ is a coupling, $v_0(> 0)$ is the Higgs field vacuum expectation value, and Λ is the 6D cosmological constant. It is later shown that the sign of Λ must be negative for the proposed pole vacuum configuration. We take the line element shown below.³

$$ds^2 = e^{-2\sigma(\rho)} \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \rho^2 e^{-2\omega(\rho)} d\varphi^2 ,$$

³ The same type metric was also taken in Ref.[18, 20].

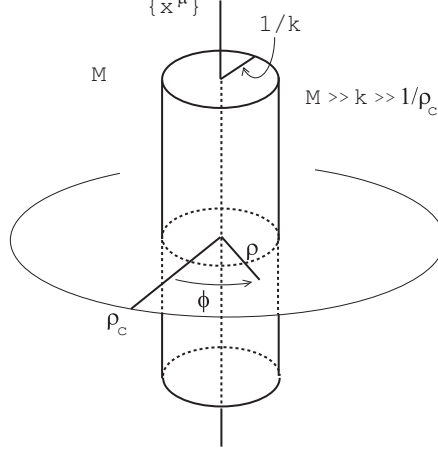


Fig.1 The pole configuration.

$$0 \leq \rho < \infty \quad , \quad 0 \leq \varphi < 2\pi \quad , \quad (2)$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. This is a natural 6D minimal-extension of the original 5D model by Randall-Sundrum [3]. For $\sigma = \omega = 0$, ds^2 is the 6D Minkowski (flat) space. In this choice, the 4D Poincaré invariance is preserved. Two "warp" factors $e^{-2\sigma(\rho)}$ and $e^{-2\omega(\rho)}$ appear and play an important role throughout this paper. Note that, for the fixed ρ case ($d\rho = 0$), the metric is the *Weyl transformation* of the product space of the flat (Minkowski) space, $\eta_{\mu\nu}dx^\mu dx^\nu$, and the circle S^1 , $\rho^2 d\varphi^2$. The two Weyl factors serve for *scaling* each space. The coordinate ρ is regarded as the scaling parameter.

From (2) we can read

$$\begin{pmatrix} G_{AB} \end{pmatrix} = \begin{pmatrix} e^{-2\sigma}\eta_{\mu\nu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 e^{-2\omega} \end{pmatrix} . \quad (3)$$

The 6D Riemannian scalar curvature \hat{R} is given by

$$\hat{R} = 4(-2\sigma'' + 5(\sigma')^2 - 2\frac{\sigma'}{\rho}) + 8\omega'\sigma' + 2(-\omega'' + (\omega')^2 - 2\frac{\omega'}{\rho}) , \quad (4)$$

where $' = \frac{d}{d\rho}$. We will later require *there are no curvature singularities anywhere* because the configuration considered has the *finite thickness* (radius

of the pole) which should smooth the curvature. Since M -dependence can be absorbed by a simple scaling ($\Phi = M^2 \tilde{\Phi}$, $v_0 = M^2 \tilde{v}_0$, $\lambda = M^{-2} \tilde{\lambda}$, $\Lambda = M^6 \tilde{\Lambda}$, $x^\mu = M^{-1} \tilde{x}^\mu$, $\rho = M^{-1} \tilde{\rho}$, $\varphi = \tilde{\varphi}$), it is sometimes useful to take

$$M = 1 \quad . \quad (5)$$

We will, however, explicitly write M as much as possible in order for the reader not to bother about the superficial dimensional mismatch.

3 A solution

Let us solve the 6D Einstein equation.

$$\begin{aligned} M^4(\hat{R}_{MN} - \frac{1}{2}G_{MN}\hat{R}) = \\ -\partial_M\Phi^*\partial_N\Phi - \partial_N\Phi^*\partial_M\Phi + G_{MN}(G^{KL}\partial_K\Phi^*\partial_L\Phi + V) \quad , \\ \nabla^2\Phi = \frac{\delta V}{\delta\Phi^*} \quad . \end{aligned} \quad (6)$$

As for the complex scalar field Φ , first of all, it should satisfy the periodicity condition: $\Phi(x, \rho, \varphi) = \Phi(x, \rho, \varphi + 2\pi)$. Following Callan and Harvey[16], we consider a simple case that Φ depends only on the extra coordinates φ and ρ as

$$\Phi_m(\rho, \varphi) = P(\rho)e^{im\varphi} \quad , \quad m = 0, \pm 1, \pm 2, \dots \quad . \quad (7)$$

Then the Einstein equation (6) reduces to

$$M = \mu, N = \nu$$

$$\begin{aligned} 3\sigma'' - 6\sigma'^2 + 3\frac{\sigma'}{\rho} - 3\sigma'\omega' + \omega'' - (\omega')^2 + 2\frac{\omega'}{\rho} \\ = M^{-4}(P'^2 + m^2\frac{e^{2\omega}}{\rho^2}P^2 + V) \quad , \end{aligned} \quad (8)$$

$$M = \rho, N = \rho$$

$$-6\sigma'^2 + 4\frac{\sigma'}{\rho} - 4\sigma'\omega' = M^{-4}(-P'^2 + m^2\frac{e^{2\omega}}{\rho^2}P^2 + V) \quad , \quad (9)$$

$$M = \varphi, N = \varphi$$

$$\text{Eq.(C)} \quad 4\sigma'' - 10\sigma'^2 = M^{-4}(P'^2 - m^2\frac{e^{2\omega}}{\rho^2}P^2 + V) \quad , \quad (10)$$

Matter Eq.

$$P'' - m^2\frac{e^{2\omega}}{\rho^2}P - (4\sigma' - \frac{1}{\rho} + \omega')P' = \frac{\lambda}{2}(P^2 - v_0^2)P \quad . \quad (11)$$

The matter equation (11) is derived from other three equations.⁴ From (8) and (9), and from (9) and (10), we obtain two convenient equations as follows.

$$\text{Eq.(A)} \quad 3\sigma'' - \frac{\sigma'}{\rho} + \sigma'\omega' + \omega'' - (\omega')^2 + 2\frac{\omega'}{\rho} = 2M^{-4}P'^2 \quad , \quad (12)$$

$$\text{Eq.(B)} \quad -16\sigma'^2 + 4\frac{\sigma'}{\rho} - 4\sigma'\omega' + 4\sigma'' = 2M^{-4}V \quad . \quad (13)$$

The three equations Eq.(A), Eq.(B) and Eq.(C) may be considered as those to be solved. They have three unknown functions $P(\rho), \sigma(\rho), \omega(\rho)$ and one unknown integer parameter m . We call this set of the three equations *reduced Einstein equations* (REE).

Now we must consider the boundary condition (asymptotic behavior), at $\rho = \infty$ (infrared region), for $P(\rho), \sigma(\rho)$ and $\omega(\rho)$. As for $P(\rho)$, we naturally take

$$\rho \rightarrow \infty \quad , \quad P(\rho) \rightarrow \pm v_0 \quad . \quad (14)$$

The plural sign \pm comes from the fact that the Higgs potential $V(\Phi)$ is the *even* function of Φ . Later soon, we will choose the plus sign above by use of the freedom of an arbitrary parameter e_0 (defined later). As for other two, we assume (from the "experience" in 5D Randall-Sundrum model[3, 4, 8]) $\sigma' \rightarrow a(\text{const}), \omega' \rightarrow b(\text{const})$ as $\rho \rightarrow \infty$. Then from the REE, we can deduce

$$\text{As } \rho \rightarrow \infty \quad , \quad \sigma' \rightarrow \tilde{\alpha} \quad , \quad \omega' \rightarrow \tilde{\alpha} \quad , \quad \tilde{\alpha} = \pm \sqrt{\frac{-\Lambda}{10}} M^{-2} \quad . \quad (15)$$

As for the plural sign above, we take the plus one which damps the scaling behavior (2) in the asymptotic region. Note that the present asymptotic requirement demands the *isotropic property* around the $\rho = 0$ axis ($m = 0$), that is, Φ should not depend on the direction φ , that is, the *pole* configuration. See Fig.1. We take the *plus sign* above in order for this mechanism to work, in the infrared region, for *exponentially suppressing* the 4D part of the metric (2). In the above result we must have the condition:

$$\Lambda < 0 \quad . \quad (16)$$

⁴ It reflects the following well-known fact. The matter equation $\nabla^2 \Phi = \frac{\delta V}{\delta \Phi^*}$ in (6) can also be derived from the (6D) energy-momentum conservation $M^4 \nabla^M (\hat{R}_{MN} - \frac{1}{2} G_{MN} \hat{R}) = \nabla^M T_{MN} = 0$, where $T_{MN} = \text{RHS of the first equation of (6)}$.

The geometry should be Anti de Sitter in the (infrared) asymptotic region. We should further note that, in the infrared region above, the 6D scalar curvature is a positive constant: $\hat{R} \rightarrow 30\tilde{\alpha}^2 = -3\Lambda M^{-4} > 0$.

Next we must consider the boundary condition at $\rho = +0$ (ultra-violet region). Assuming the following things: 1) As $\rho \rightarrow +0$, the three unknown functions σ', ω' and P goes like $\sigma' \rightarrow s\rho^a$, $\omega' \rightarrow w\rho^b$, $P \rightarrow x\rho^c$ where s, w and x are *non-zero* constants ($s \neq 0, w \neq 0, x \neq 0$); 2) The 6D scalar curvature \hat{R} is *regular everywhere*; 3)⁵ $a = b$, we can deduce, from REE, that there are two cases:

$$\text{Case 1} \quad a = b = 1, \quad c = 0, \quad s = \frac{\lambda}{16}(x^2 - v_0^2)^2 + \frac{\Lambda}{4} \quad (17)$$

$$\text{Case 2} \quad a = b = 1, \quad c > 1, \quad s = \frac{\lambda}{16}v_0^4 + \frac{\Lambda}{4} \quad (18)$$

If we put off the condition $x \neq 0$ in (17) and take $x = 0$, eq.(17) reduces to (18).

Let us take the following form for $\sigma'(\rho), \omega'(\rho)$ and $P(\rho)$ as a solution ⁶.

$$\begin{aligned} \sigma'(\rho) &= \alpha \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+1)!} \{\tanh(k\rho)\}^{2n+1}, \\ \omega'(\rho) &= \alpha \sum_{n=0}^{\infty} \frac{d_{2n+1}}{(2n+1)!} \{\tanh(k\rho)\}^{2n+1}, \\ P(\rho) &= v_0 \sum_{n=0}^{\infty} \frac{e_{2n}}{(2n)!} \{\tanh(k\rho)\}^{2n}, \end{aligned} \quad (19)$$

where $\alpha \equiv +\sqrt{\frac{-\Lambda}{10}}M^{-2}$ and c 's, d 's and e 's are coefficient-constants (independent of ρ) to be determined. Because REE have the "formal" symmetry of *parity*: $\rho \rightarrow -\rho$, we know the solutions are parity odd or even functions. (Note that $0 \leq \rho < \infty$, (2).) In (19), σ' and ω' are composed of *odd* powers of $\tanh(k\rho)$, whereas P is of *even* powers. The choice is deduced by the behavior at the ultra-violet region (17,18). The expansion (19) could be regarded as a sort of "Kaluza-Klein mode expansion". ⁷ In the present

⁵ The assumption $a = b$ is based on the expected "duality" between IR and UV behaviors. Note that, in the IR, σ' and ω' behave similarly in (15).

⁶ Some properties of the form of the series (19) in relation to the important equation (41) is described in App.A.

⁷ In comparison with the work by Goldberger and Wise[22], not only the scalar field P but also the warp factors σ' and ω' are "Kaluza-Klein expanded".

case, however, there is *no periodicity* in the ρ -coordinate. We *neither* have the *translation invariance* $\rho \rightarrow \rho + \text{const.}$, which should be compared with 5D case[8, 9]. A *new mass scale* $k(> 0)$ is introduced here to make the quantity $k\rho$ dimensionless. The physical meaning of $1/k$ is the "thickness" of the pole. The parameter k , with M and ρ_c (defined later), plays a central role in this dimensional reduction scenario. We call them, (k, M, ρ_c) , *fundamental parameters*. The distortion of 6D space-time by the existence of the pole should be sufficiently small so that the quantum effect of 6D gravity can be ignored and the present *classical* analysis is valid. This requires the condition[3]

$$k \ll M \quad . \quad (20)$$

The boundary conditions (14,15) with the + choice in the plural signs, require the coefficient-constants to have the following constraints

$$1 = \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+1)!} \quad , \quad 1 = \sum_{n=0}^{\infty} \frac{d_{2n+1}}{(2n+1)!} \quad , \quad 1 = \sum_{n=0}^{\infty} \frac{e_{2n}}{(2n)!} \quad , \quad (21)$$

, which are obtained by considering the asymptotic behaviors $\rho \rightarrow \infty$ in (19). We will use these constraints in Sec.6. In App.A, we explain the determination of all coefficients in (19) from the REE. All ones are fixed except one *free* parameter $e_0(= P(0)/v_0)$.⁸ The first two orders are concretely given as

$$\begin{cases} c_1 = \frac{M^{-4}}{4k\alpha} \left\{ \frac{1}{4} \lambda v_0^4 (1 - e_0^2)^2 + \Lambda \right\} \quad , \\ d_1 = -\frac{2}{3} c_1 \quad , \\ e_0 : \text{free parameter}, \end{cases} \quad \begin{cases} c_3 = \frac{3}{32} \frac{\lambda^2 v_0^6}{k^3 \alpha} M^{-4} e_0^2 (1 - e_0^2)^2 + c_1 (2 + 5 \frac{\alpha}{k} c_1) \quad , \\ d_3 = -\frac{4}{3} c_1 (1 + 5 \frac{\alpha}{k} c_1) \quad , \\ e_2 = -\frac{1}{4} \frac{\lambda v_0^2}{k^2} e_0 (1 - e_0^2) \quad , \end{cases} \quad (22)$$

Here we note that those parts of the UV boundary condition (17,18), which are not taken into account in the starting assumption (19), are satisfied by

⁸ The corresponding *global* symmetry looks some 'deformation' of the constant transformation: $P(\rho) \rightarrow P(\rho) + e_0$. The symmetry is valid for the special case: $m = 0, \lambda = 0$. This situation should be compared with the 5D case where the constant translation symmetry for a coordinate exists[8, 9].

the above solution: [Case 1] $e_0 \neq 0, x = v_0 e_0, s = \alpha c_1$; [Case2] $e_0 = 0, s = \alpha c_1$. The third order is given by

$$\begin{aligned}
& e_4 \times M^{-4} \left\{ \frac{kv_0^2}{\alpha} e_2 + \frac{1}{12} \frac{\lambda v_0^4}{k\alpha} e_0 (1 - e_0^2) \right\} \\
& = \left(-\frac{4}{9} c_3 + 2c_1 \right) + \frac{\alpha}{k} \cdot \frac{5}{3} (d_1 c_3 + d_3 c_1 + 2c_1 c_3) \\
& \quad - \frac{\lambda v_0^4}{k\alpha} \frac{1}{4} M^{-4} (1 - 3e_0^2) e_2^2 + 6 \frac{kv_0^2}{\alpha} M^{-4} e_2^2, \\
& c_5 = -\frac{5}{12} \frac{\lambda v_0^4}{k\alpha} M^{-4} e_0 (1 - e_0^2) e_4 + \frac{100}{9} c_3 + 4c_1 \\
& \quad + \frac{\alpha}{k} \cdot \frac{10}{3} (d_1 c_3 + d_3 c_1 + 8c_1 c_3) - \frac{5}{4} \frac{\lambda v_0^4}{k\alpha} M^{-4} (1 - 3e_0^2) e_2^2, \\
& d_5 = -2c_5 + \frac{120}{7} \left\{ \frac{2}{3} \frac{kv_0^2}{\alpha} M^{-4} e_2 e_4 + \frac{13}{9} c_3 - \frac{1}{5} c_1 + \frac{11}{18} d_3 + \frac{2}{5} d_1 \right. \\
& \quad \left. - \frac{\alpha}{k} \cdot \frac{1}{6} (d_1 c_3 + d_3 c_1 - 2d_1 d_3) - 4 \frac{kv_0^2}{\alpha} M^{-4} e_2^2 \right\} \quad (23)
\end{aligned}$$

Note that, in the first equation, e_4 is written by lower-order terms ($e_0, e_2; c_1, c_3; d_1, d_3$). Putting this result of e_4 into the second equation, we see c_5 is written by the lower-order terms. Similarly to the third equation with respect to d_5 .

The general terms $(c_{2n+1}, d_{2n+1}, e_{2n}), n \geq 3$ are obtained in App.A. They are expressed by the coupled *linear* equations (57) with respect to them and can be solved recursively in the increasing order of n . All coefficients are expressed by four parameters λ, v_0, Λ and e_0 . (We may take $M = 1$ as explained before. We may also take $k = 1$, keeping the relation (20) in mind. This is because the thickness parameter k can be absorbed by the change : $c_{2n+1} \rightarrow \frac{\alpha}{k} c_{2n+1}, d_{2n+1} \rightarrow \frac{\alpha}{k} d_{2n+1}, \Lambda \rightarrow \frac{\Lambda}{k^2} M^{-4}, \lambda \rightarrow \frac{\lambda}{k^2} M^4$. We will, however, keep k as much as possible in order for the reader not to bother about the superficial dimensional mismatch.) In the above results, we see $e_{2n} (n \geq 1)$ is the odd function of e_0 , whereas c_{2n+1} and d_{2n+1} are the even functions. Under the sign change $e_0 \rightarrow -e_0$, P changes the sign, whereas σ' and ω' do not. As commented before, we can choose the plus sign in (14) by use of this freedom. The above four parameters have three constraints (21) from the boundary condition at the infrared infinity. Hence the present solution is *one-parameter family* solution. We will solve the constraints in Sec.6. As a final note in this section, we point out that, in comparison with 5D RS-model[8, 9], there appears no lower-bound for Λ .

4 Meaning of the Solution

Let us examine the obtained solution in relation to the two general theorems: Goldstone's theorem and the Derrick's theorem.

(i) Spontaneous Symmetry Breakdown

The present model (1) has the discrete symmetry:

$$\Phi \rightarrow -\Phi, \quad \Phi^* \rightarrow -\Phi^*, \quad (24)$$

and the continuous (global) symmetry:

$$\Phi \rightarrow e^{i\theta}\Phi, \quad \Phi^* \rightarrow e^{-i\theta}\Phi^*, \quad (25)$$

where θ is a constant parameter. (24) is a special case of (25) and appears also in the case of the real scalar field. If the vacuum breaks the continuous symmetry, a massless scalar boson appears (Goldstone's theorem[23, 24]). In the present case, however, we confine the complex scalar field Φ to the real field by taking only the $m = 0$ mode in (7). (In fact we can start from the real scalar Φ in (1) and obtain the same result.) This is required from the demand that the IR boundary condition of the warp factors is the same as the 5D RS-model. Hence we should understand that, in the present solution, the discrete symmetry (24) is spontaneously broken, not the continuous one (25). No Goldstone boson appears, and the situation is similar to the domain wall case of 5D model [8, 9]. When we want to consider the breaking of the continuous symmetry, the standard way is to take the Abelian Higgs model[25] where the Goldstone boson is 'eaten'⁹ by the gauge field (Higgs phenomenon). Other approach is also examined in [26].

(ii) Role of the Warp Factors

After taking the 'warped' metric (2) and assuming the form of the complex scalar field (7) with $m = 0$, the action reduces to

$$S = 2\pi \int d^4x \, d\rho \, \rho e^{-4\sigma-\omega} \{ e^{2\sigma} \dot{\Phi}^* \dot{\Phi} - P'^2 + 2(2\sigma'' - 5\sigma'^2 + 2\frac{\sigma'}{\rho}) - 4\omega'\sigma' + (\omega'' - \omega'^2 + 2\frac{\omega'}{\rho}) - \frac{\lambda}{4}(P^2 - v_0^2)^2 - \Lambda \} \quad (26)$$

where $\dot{\Phi} = \frac{d\Phi}{dt}$ which vanishes in the present static solution. If we omit the kinetic term of $\dot{\Phi}^* \dot{\Phi}$ in the above integral formula, the remaining part is

⁹ The massless Goldstone mode is replaced by one mode of the massive gauge field.

regarded as $(-1) \times (\text{Energy of the static solution})$. Its finiteness is apparent from the factor of $e^{-4\sigma-\omega}$ and the regular behavior of P, σ' and ω' near $\rho = 0$. The expression (26) looks like an interacting theory of scalars. As for the localized solutions in scalar theories, there exists the well-known theorem called Derrick's theorem[27, 24]. It says, in the case of the space dimensions higher than or equal to 2, the only non-singular time-independent solutions of finite energy are the ground states. The present case escapes this trivial situation in the following way. As shown in (26), the interaction forms of the fields $\sigma(\rho), \omega(\rho)$ are not those of scalars but those of dilatons. Especially the linear terms $(\sigma'', \sigma'/\rho, \omega'', \omega'/\rho)$ appear. They break the positivity property used in the proof of the Derrick's theorem.

5 Vacuum parameters: M and k -dependence in the dimensional reduction

Let us examine the behavior of the vacuum parameters (Λ, v_0, λ) near the 4D world limit (thin pole limit): $k \rightarrow \infty$ (the dimensional reduction). This should be taken consistently with (20). We will specify the above limit in the more well-defined way later. In (19), we note that $\{\tanh(k\rho)\}^{2n+1} \rightarrow \theta(k\rho)$, $\{\tanh(k\rho)\}^{2n} \rightarrow \theta(k\rho)$, $\rho \geq 0$ as $k \rightarrow \infty$. Assuming the convergence of the infinite series, we can conclude that $c_1 \sim O(k^0)$, $e_2 \sim O(k^0)$ as $k \rightarrow \infty$. (e_0 is a free parameter.) From the former relation, using the explicit result of c_1 (22), we know $-\Lambda \sim \lambda v_0^4 \sim M^4 k \alpha$. Because $\alpha \sim \sqrt{-\Lambda} M^{-2}$, this says $\sqrt{-\Lambda} \sim k M^2$ and $\lambda v_0^4 \sim k^2 M^4$. From the latter one, using the explicit result of e_2 (22), we know $\lambda v_0^2 \sim k^2$. Hence we obtain

$$-\Lambda \sim M^4 k^2, \quad v_0 \sim M^2, \quad \lambda \sim M^{-4} k^2 \quad \text{as } k \rightarrow \infty. \quad (27)$$

These are the *leading* behavior of the vacuum parameters in the dimensional reduction from 6D to 4D as $k \rightarrow \infty$. As for the k -dependence, the above result is the *same* as the 5D model of Randall and Sundrum[3, 8]. The more precise forms of (27) will be obtained, in Sec.6, using the constraints (21).

6 Physical constants and See-Saw relation

In order to express some physical scales in terms of the fundamental parameters M, k and ρ_c (to be introduced soon), we consider the case that the 4D

geometry is slightly fluctuating around the Minkowski (flat) space.

$$\begin{aligned} ds^2 &= e^{-2\sigma(\rho)} g_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \rho^2 e^{-2\omega(\rho)} d\varphi^2 \quad , \\ g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad h_{\mu\nu} \sim O\left(\frac{1}{k}\right) \quad . \end{aligned} \quad (28)$$

The leading order $O(k^0)$ results of the previous section remain valid.

6.1 The Planck mass

The gravitational part of 6D action (1) reduces to 4D action as

$$\int d^6 X \sqrt{-G} M^4 \hat{R} \sim M^4 \int_0^{2\pi} d\varphi \int_0^{\rho_c} d\rho \rho e^{-2\sigma(\rho)-\omega(\rho)} \int d^4 x \sqrt{-g} R + \dots \quad , (29)$$

where the *infrared regularization* parameter ρ_c is introduced. ρ_c specifies the *size* of the extra 2D space. Using the asymptotic forms, $\sigma \sim \alpha\rho$, $\omega \sim \alpha\rho$ as $\rho \rightarrow \infty$ and $\alpha = \sqrt{\frac{-\Lambda}{10}} M^{-2} \sim k$ as $k \rightarrow \infty$, we can evaluate the order of M_{pl} as¹⁰

$$M_{pl}^2 \sim M^4 2\pi \int_0^{\rho_c} d\rho \rho e^{-3\alpha\rho} \sim \frac{M^4}{\alpha^2} \sim \frac{M^4}{k^2} \quad , \quad (30)$$

where we have used the *4D reduction condition*:

$$k\rho_c \gg 1 \quad . \quad (31)$$

The result (30) is *different* from 5D model of Randall-Sunsum [3, 8]: $M_{pl}^2 \sim \frac{M^3}{k}$.

The above condition (31) should be interpreted as the precise (well regularized) definition of $k \rightarrow \infty$ (dimensional reduction) used so far. We note ρ_c dependence in (30) is negligible for $k\rho_c \gg 1$. This behavior shows the distinguished contrast with the Kaluza-Klein reduction of 6D to 4D ($M_{pl}^2 \sim M^4(\rho_c)^2$) as stressed in [3] for the 5D case.

Writing (30) as

$$\frac{M_{pl}}{M} \sim \frac{M}{k} \quad , \quad (32)$$

we notice this mass relation is the *geometrical see-saw* relation corresponding to the matrix :

$$\begin{pmatrix} 0 & M \\ M & M_{pl} \end{pmatrix} \quad . \quad (33)$$

¹⁰ A similar relation is derived in other 6D model[20]

This provides the geometrical approach to the see-saw mechanism [28, 29] which is usually explained by the diagonalization of the (neutrino) mass matrix. (See a textbook[30].) For some discussion about the neutrino mass, using the above relation, see the final section.

6.2 The cosmological term

The cosmological part of (1) reduces to 4D action as

$$\int d^6 X \sqrt{-G} \Lambda \sim \Lambda \int_0^{2\pi} d\varphi \int_0^{\rho_c} d\rho \rho e^{-4\sigma(\rho) - \omega(\rho)} \int d^4 x \sqrt{-g} \equiv \Lambda_{4d} \int d^4 x \sqrt{-g} \quad ,$$

$$\Lambda_{4d} \sim \frac{\Lambda}{\alpha^2} \sim -M^4 < 0 \quad , \quad kr_c \gg 1 \quad .(34)$$

Λ_{4d} is the cosmological term in the 4D space-time. It does not, like M_{pl} , depend on ρ_c strongly. The above result differs from RS-model in that k -dependence disappears. (cf. $\Lambda_{4d} \sim -M^3 k$ in RS-model[8].) The result says the 4D space-time should also be *anti de Sitter*.

6.3 Numerical fitting of M, k and ρ_c

Let us examine what orders of values should we take for the fundamental parameters M and k . (ρ_c is later fixed by the information of the 4D fermion masses.) Using the value $M_{pl} \sim 10^{19} \text{GeV}$, the "rescaled" cosmological parameter $\tilde{\Lambda}_{4d} \equiv \Lambda_{4d}/M_{pl}^2$ [31] has the relation:

$$\sqrt{-\tilde{\Lambda}_{4d}} \sim k \sim M^2 \times 10^{-19} \text{ GeV} \quad , \quad (35)$$

where the relations (30) and (34) are used. The unit of mass is GeV and this unit is taken in the following. The observed value of $\tilde{\Lambda}_{4d}$ is not definite, even for its sign.[32] If we take into account the quantum effect, the value of $\tilde{\Lambda}_{4d}$ could run along the renormalization[33]. Furthermore if we consider the parameter $\tilde{\Lambda}_{4d}$ represents some "effective" value averaging over all matter fields, the value, no doubt, changes during the evolution of the universe. (Note the model (1) has no (ordinary) matter fields.) Therefore, instead of specifying $\tilde{\Lambda}_{4d}$, it is useful to consider various possible cases of $\tilde{\Lambda}_{4d} \sim -k^2$.

Some typical cases are 1) ($k = 10^{-41}, M = 10^{-11}$), 2) ($k = 10^{-13}, M = 10^3$) 3) ($k = 10, M = 10^{10}$) 4) ($k = 10^4, M = 10^{11.5}$) and 5) ($k = 10^{19}, M = 10^{19}$). Case 1) gives the most plausible present value of the cosmological constant. The wall thickness $1/k = 10^{41} [\text{GeV}^{-1}]$, however, is the radius of

the present universe. This implies the extra dimensional effect appears at the cosmological scale, which should be abandoned in the usual sense. For further discussion, see the final section. Case 2) gives $1/k = 10^{13} \text{ GeV}^{-1} \sim 1\text{mm}$ which is the minimum length at which the Newton's law is checked[34, 35]. Usually k should be larger than this value so that we keep the observed Newton's law. 5) is an extreme case $M = M_{pl}$. The fundamental scale M is given by the Planck mass. In this case, $\rho_c \gg 1/k = 1/M_{pl}$ is acceptable, whereas $\sqrt{-\tilde{\Lambda}_{4d}} \sim M_{pl}$ is completely inconsistent with the experiment and requires explanation. Most crucially the condition (20) breaks down. Cases 3) and 4) are some intermediate cases which are acceptable except for the cosmological constant. They will be used in the next paragraph. At present any choice of (k, M) looks to have some trouble if we take into account the cosmological constant. We consider the observed cosmological constant (10^{-41}GeV) should be explained by some unknown mechanism.

As in the Callan and Harvey's paper[16], we can have the 4D *massless chiral* fermion bound to the wall by introducing 6D *Dirac* fermion ψ into (1).

$$S[G_{AB}, \Phi] + \int d^6 X \sqrt{-G} (\bar{\psi} \not{\nabla} \psi + g \bar{\psi} (\text{Re}\Phi - i\Gamma^7 \text{Im}\Phi) \psi) \quad . \quad (36)$$

If we *regulate* the extra axis by the finite range $0 \leq y \leq \rho_c$, the 4D fermion is expected to have a small mass $m_f \sim k e^{-k\rho_c}$ (This phenomenon is known in the condensed matter physics as the surface mode. It is also confirmed in the two-walls configuration of the 5D lattice theory [36, 37]). If we take the case 3) in Subsec.5.3 ($k = 10, M = 10^{10}$) and regard the 4D fermion as a neutrino ($m_\nu \sim 10^{-11} - 10^{-9}\text{GeV}$), we obtain $\rho_c = 2.76 - 2.30\text{GeV}^{-1}$. If we take case 4) ($k = 10^4, M = 10^{11.5}$), we obtain $\rho_c = (3.45 - 2.99) \times 10^{-3}\text{GeV}^{-1}$. When the quarks or other leptons ($m_q, m_l \sim 10^{-3} - 10^2\text{GeV}$) are taken as the 4D fermion, and take the case 4) in Subsec.5.3, we obtain $\rho_c = (1.61 - 0.461) \times 10^{-3}\text{GeV}^{-1}$. It is quite a fascinating idea to *identify the chiral fermion zero mode bound to the pole with the neutrinos, quarks or other leptons*.

6.4 The valid region of the present approach

The condition on k in the present model, from (20) and (31), is given as

$$\frac{1}{\rho_c} \ll k \ll M \quad . \quad (37)$$

In ref.[8], it is pointed out that the present mechanism has the similarity to the chiral fermion determinant. The above relation corresponds to $|k^\mu| \ll M_F \ll \frac{1}{t}$ in ref.[38, 39]. Especially the thickness parameter k corresponds to 5D fermion mass parameter M_F which is a very important parameter, in the lattice simulation, to give good numerical results for the soft-hadron physics (such as the pion mass).

7 Precise form of Solution (Analytical Approach)

Let us determine the precise form of the solution by solving the boundary condition (21). An interesting aspect of the present solution is that some family of vacua is selected as the consistent (classical) configuration. In other words, the vacuum parameters are fixed to be some numbers (if one free parameter is fixed). Let us determine the precise form of (27) and (19) using the three constraints (21). The coefficients $(c_{2n+1}, d_{2n+1}, e_{2n})$ of (19) are solved, in (22) and (23), in terms of four parameters λ, v_0, Λ and e_0 . Therefore the present solution is *one parameter* family solution. The precise forms are obtained by the $\frac{1}{k^2}$ -expansion for the case $k\rho_c \gg 1$ as

$$\begin{aligned}\sqrt{-\Lambda} &= M^2 k (\alpha_0 + \frac{\alpha_1}{(k\rho_c)^2} + \cdots) = M^2 k \sum_{n=0}^{\infty} \frac{\alpha_n}{(k\rho_c)^{2n}} \quad , \\ \lambda v_0^4 &= M^4 k^2 (\delta_0 + \frac{\delta_1}{(k\rho_c)^2} + \cdots) = M^4 k^2 \sum_{n=0}^{\infty} \frac{\delta_n}{(k\rho_c)^{2n}} \quad , \\ v_0^2 &= M^4 (\beta_0 + \frac{\beta_1}{(k\rho_c)^2} + \cdots) = M^4 \sum_{n=0}^{\infty} \frac{\beta_n}{(k\rho_c)^{2n}} \quad ,\end{aligned}\tag{38}$$

where α 's, δ 's and β 's are some numerical (real) numbers to be consistently chosen using (21).

If we assume the infinite series of (21) converge sufficiently rapidly, we can safely truncate it at the first few terms. In order to demonstrate how the vacuum parameters are fixed, we take into account up to $n = 2$ in (21) and the leading order in (38).¹¹ We present two sample solutions:

$$\begin{aligned}\text{Solution 1 } e_0 &= -0.5(\text{input}), \\ (\alpha_0, \beta_0, \delta_0) &= (2.507019, 1.113947, 29.14319) \quad ;\end{aligned}$$

¹¹ In ref.[9], the improved calculation results are obtained. It takes into account terms up to $n=6$ th order.

$$\begin{aligned}
& \text{Solution 2 } e_0 = -0.8(\text{input}), \\
& (\alpha_0, \beta_0, \delta_0) = (1.864786, 0.6424929, 15.90691).
\end{aligned} \tag{39}$$

For each solution, the expansion coefficients ¹² are obtained as

Solution 1

$$\begin{aligned}
(c_1, c_3, c_5) &= (-0.689617, 13.1849, -60.9444), \\
(d_1, d_3, d_5) &= (0.459745, -1.59403, 96.7112), \\
(e_0, e_2, e_4) &= (-0.5, 2.45270, 6.56764);
\end{aligned}$$

Solution 2

$$\begin{aligned}
(c_1, c_3, c_5) &= (-1.25575, 7.33112, 124.067), \\
(d_1, d_3, d_5) &= (0.837165, -4.52497, 110.039), \\
(e_0, e_2, e_4) &= (-0.8, 1.78258, 21.809).
\end{aligned} \tag{40}$$

For the solution 2, we plot $P(\rho)$, $\sigma'(\rho)$, $\omega'(\rho)$ and $\hat{R}(\rho)$ in Fig.2, Fig.3, Fig.4 and Fig.5, respectively. We stress that the Riemann scalar curvature \hat{R} is *everywhere non-singular*. Note that $\hat{R}/\alpha^2 \rightarrow -12\sqrt{10}c_1/A$, $A \equiv \sqrt{-\Lambda}/(M^2k)$ as $\rho \rightarrow +0$ (ultraviolet region). (Purely for the technical reason, Fig.5 do not show the value of \hat{R} at $\rho = 0$.) As for the infrared region, we know $\hat{R}/\alpha^2 \rightarrow 30$ as $\rho \rightarrow \infty$.

Note the following points: 1) The above results are valid for general fundamental parameters (k, M, ρ_c) except the thin-pole condition $k\rho_c \gg 1$; 2) For each given value of e_0 , the solution looks unique (when a solution exists) as far as the reasonable range of $(\alpha_0, \beta_0, \delta_0)$ is concerned; 3) For the choice of the plus boundary for the Higgs field P (the plus sign in (14)), we do not find solutions for positive e_0 in the reasonable range of parameters; 4) For one given input value (e_0 in the above case. The 6D Higgs mass (in unit k) $\sqrt{\lambda v_0^2}/k \sim \sqrt{\delta_0/\beta_0}$ is another example of such value.) the cosmological constant (in unit M^2k) $\sqrt{-\Lambda}/M^2k \sim \alpha_0$ is fixed ¹³; 5) The above very

¹² When we see the convergence behavior of the series appearing in (19) and (21), we should take into account the factorial factors which divide the all coefficients c's, d's and e's.

¹³ In some references, this is mistakenly called "fine-tuning". The cosmological constant is here not fine-tuned but fixed from the (classical) field equation and the input value.

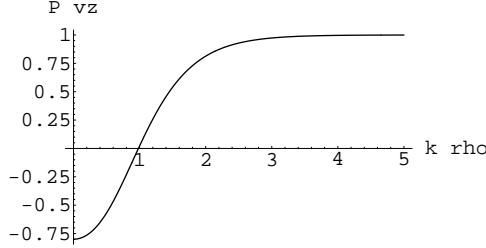


Fig.2 The Higgs Field $P(\rho)/v_0$, (19). Solution 2 in (39) and (40).
Horizontal axis: $k\rho$.

fine results are obtained by the standard method of the numerical calculus:
Newton method.

Our solution has one *free parameter*. Using this freedom we can adjust one of the three vacuum parameters in the way the observed physical values are explained. In (39), we take $e_0 = P(0)/v_0$ as the adjustment. Solution 2 has the vacuum expectation value $v_0 M^{-2} \sim \sqrt{\beta_0} = 0.802$, the cosmological constant $\Lambda M^{-4} k^{-2} \sim -\alpha_0^2 = -3.48$ and the 6D Higgs mass $\sqrt{\lambda v_0^2} k^{-1} \sim \sqrt{\delta_0/\beta_0} = 4.98$. We notice the dimensionless vacuum parameters, $v_0 M^{-2}$, $\Lambda k^{-2} M^{-4}$ and $\lambda k^{-2} M^4$, are specified only by the value of $k\rho_c$ and one input data, say, e_0 . When we further specify k and M , as considered in Subsec 5.3, the values v_0 , Λ and λ are themselves obtained. Any higher-order, in principle, can be obtained by the $\frac{1}{k^2}$ -expansion.

8 Numerical Solution

Instead of the previous analytic approach based on the assumed function form (19), we can directly solve the coupled differential equations Eq.(A), Eq.(B) and Eq.(C) ((12),(13) and (10)) in the numerical method (Runge-Kutta method). The difficult point of this approach is the right choice of the vacuum parameters. The three parameters should be correctly chosen corresponding to a given boundary condition. (In the previous section, say, the condition is the initial value of P : $P(\rho=0)/v_0 = e_0$.) Another difficulty comes from the technical reason in the numerical analysis. In the coupled equation, there exists the "superficial" singularity at $\rho = 0$. (Of course, there

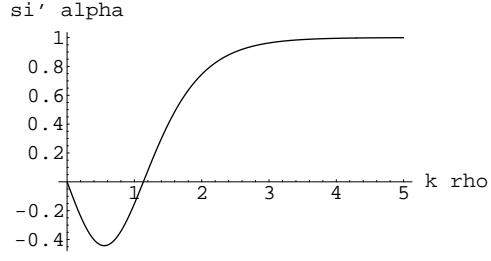


Fig.3 The warp factor $\sigma'(\rho)/\alpha$, (19). Solution 2 in (39) and (40).
Horizontal axis: $k\rho$.

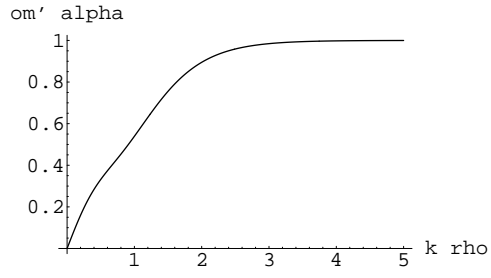


Fig.4 The warp factor $\omega'(\rho)/\alpha$, (19). Solution 2 in (39) and (40).
Horizontal axis: $k\rho$.

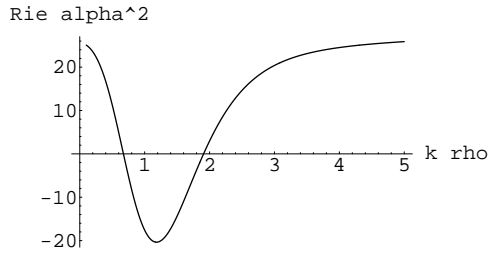


Fig.5 6D Riemann Scalar Curvature $\hat{R}(\rho)/\alpha^2$, (4). Solution 2 in (39) and (40). Horizontal axis: $k\rho$.

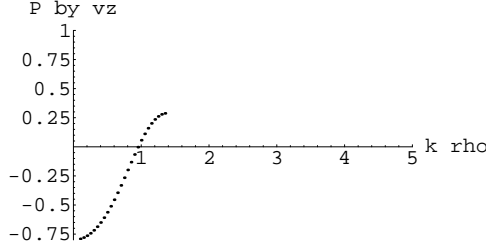


Fig.6 The Higgs Field $P(\rho)/v_0$ obtained by the Runge-Kutta method.
Ultra start. Horizontal axis: $k\rho$.

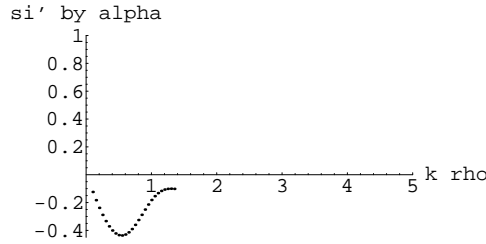


Fig.7 The warp factor $\sigma'(\rho)/\alpha$ obtained by the Runge-Kutta method.
Ultra start. Horizontal axis: $k\rho$.

is no singularity in the obtained solution.) To avoid the singular point, we must start, not from $\rho = 0$ but from a point slightly away from $\rho = 0$ in the positive direction, say, $k\rho = 0.1$. In this case, we must properly choose the initial values of $P(k\rho = 0.1)$, $\sigma'(k\rho = 0.1)$ and $\omega'(k\rho = 0.1)$. For these values (the vacuum parameters and the initial values) we can borrow the information from the analytic result in the previous section.¹⁴ In Fig.6,7, and 8, we plot the results for $P(\rho)$, $\sigma'(\rho)$ and $\omega'(\rho)$ respectively.

The numerical results well reproduce the solution of the previous analytic approach in the ultra-violet region. (Note that we do not assume the form of the solution (19).) The calculation, however, breaks down at $k\rho \approx 1$. This

¹⁴ See App.B (i)Ultra-Violet Start for detail.

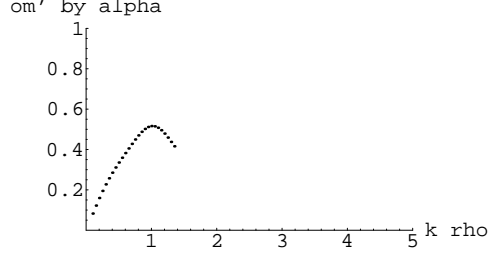


Fig.8 The warp factor $\omega'(\rho)/\alpha$ obtained by the Runge-Kutta method.
Ultra start. Horizontal axis: $k\rho$.

is because the insufficiency of the correct choice of the parameters and the initial values. We already know how finely those quantities should be chosen, in the 5D case, in order to obtain the required boundary condition[10]. See Appendix B for detail, where the results of the infra-start are also presented.

Note that the fact that the cosmological constant and other parameters must be "fine-tuned" to give the required boundary condition, just means those are fixed to be some definite values. It is similar to the situation the energy eigen values of the hydrogen atom are fixed by the boundary condition. It does *never* mean unstableness of the solution.

9 Discussion and conclusion

We summarize the present result by comparing it with the case of 5D model by Randall and Sundrum.

	5 Dim.(Randall-Sundrum)	6 Dim.
Dimensions of Vacuum Parameters	$[v_0]=(\text{mass})^{3/2}$ $[\lambda]=(\text{mass})^{-1}$ $[\Lambda]=(\text{mass})^5$	$[v_0]=(\text{mass})^2$ $[\lambda]=(\text{mass})^{-2}$ $[\Lambda]=(\text{mass})^6$
Cond. on Cosm. Term	$-\frac{\lambda v_0^4}{4} < \Lambda < 0$ Lower Bound exists	$\Lambda < 0$ No Lower Bound
Cond. on Fund. Parameters	$\frac{1}{r_c} \ll k \ll M$	$\frac{1}{\rho_c} \ll k \ll M$
Asym. Behav. of Vac. Parameters (same k -dep.)	$-\Lambda \sim M^3 k^2$ $v_0 \sim M^{3/2}$ $\lambda \sim M^{-3} k^2$ $\sqrt{\lambda v_0^2} \sim k$	$-\Lambda \sim M^4 k^2$ $v_0 \sim M^2$ $\lambda \sim M^{-4} k^2$ $\sqrt{\lambda v_0^2} \sim k$
Mass Relations	$\frac{M_{pl}}{M} \sim \sqrt{\frac{M}{k}}$ $\Lambda_{4d} = M_{pl}^2 \tilde{\Lambda}_{4d} \sim -M^3 k$	$\frac{M_{pl}}{M} \sim \frac{M}{k}$ see-saw rel. $\Lambda_{4d} = M_{pl}^2 \tilde{\Lambda}_{4d} \sim -M^4$
4D Cosm. Term	$\sqrt{-\tilde{\Lambda}_{4d}} \sim k \sim M^3 \times 10^{-38} \text{GeV}$	$\sqrt{-\tilde{\Lambda}_{4d}} \sim k \sim M^2 \times 10^{-19} \text{GeV}$

Table 1 Comparison of 5D model and 6D model.

We add some numerical fact about the see-saw relation (32). If we take as $k \sim 10^{-41} \text{GeV}$ (inverse of the cosmological size), we obtain $M \sim 10^{-11} \text{GeV} = 10^{-2} \text{eV}$ which is the order of the neutrino mass. This is the case 1) in the second paragraph of Subsec.5.3, which was abandoned there.¹⁵ This choice is, however, attractive in that it gives the right value of the cosmological constant. If this numerical fact is not accidental and has meaning, it says the cosmological size is related to the neutrino mass when it is "see-sawed" with the Planck mass. These three fundamental scales are geometrically

¹⁵ This choice means, as referred in subsec.5.3, the space-time behaves as six dimensional at the cosmological scale. In this connection, some interesting discussions are made in [40, 41]. They examine some scenario where the world looks like D-dimensions with $D \geq 5$, above a very large (cosmological) size.

related. The 6D fundamental scale M gives the mass value, and which suggests the neutrino mass is radiatively generated in 6D *quantum* gravity. We consider this is an interesting mechanism to explain the fact that, in the log-scale plot of the characteristic mass scales, the neutrino mass is located at the middle of the two scales: the Planck scale(10^{19}GeV) and the cosmological constant scale(10^{-41}GeV). Of course, in order to confidently accept this view, we should solve the problem presented in Subsec.5.3.

One of the fundamental parameters, ρ_c , is introduced in Subsec.5.1 as the infrared regularization. This is quite natural in the standpoint of the discretized approach such as the lattice. The treatment, however, should be regarded as an "effective" approach or a "temporary" stage of a right treatment. One proposal is made in [10] for the 5D case where a new infrared regularization is introduced. The metric (2) does not have any singularity or horizon in the region $0 \leq \rho < \infty$. $\rho = \infty$ can be regarded as a horizon. The deficit angle there is 2π .

We hope the present result of the hierarchy model will lead to the bridge between the field theory of the soliton physics and the string field theory of the D-brane physics.

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10 Appendix A: Calculation of Coefficients

In this appendix we solve the REE ($M = 1$) of Sec.3:

$$\text{Eq.(A)} \quad 3\sigma'' - \frac{\sigma'}{\rho} + \sigma'\omega' + \omega'' - (\omega')^2 + 2\frac{\omega'}{\rho} = 2P'^2, \quad (12)$$

$$\text{Eq. (B)} \quad -8\sigma'^2 + 2\frac{\sigma'}{\rho} - 2\sigma'\omega' + 2\sigma'' = V \quad , (13)$$

$$\text{Eq. (C)} \quad 4\sigma'' - 10\sigma'^2 = (P'^2 - m^2 \frac{e^{2\omega}}{\rho^2} P^2 + V) \quad , (10) \quad ,$$

in the following form:

$$\begin{aligned} \sigma'(\rho) &= \alpha \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+1)!} \{\tanh(k\rho)\}^{2n+1} \quad , \\ \omega'(\rho) &= \alpha \sum_{n=0}^{\infty} \frac{d_{2n+1}}{(2n+1)!} \{\tanh(k\rho)\}^{2n+1} \quad , \\ P(\rho) &= v_0 \sum_{n=0}^{\infty} \frac{e_{2n}}{(2n)!} \{\tanh(k\rho)\}^{2n} \quad , (19) \end{aligned}$$

where $\alpha = \sqrt{-\Lambda/10}M^{-2}$, (15), and $V = (\lambda/4)(P^2 - v_0^2)^2 + \Lambda$, (1). λ, v_0 and Λ are some constants called *vacuum parameters* in the text. $1/k$ is another parameter called *thickness* parameter. The integer parameter m in Eq.(C) is taken to be $m = 0$ as explained in (15). The coefficient-constants c 's, d 's and e 's are to be determined. $1/\rho$, appearing above, can be expressed as

$$\begin{aligned} \frac{1}{\rho} &= \frac{2k}{\tanh(k\rho)} \sum_{n=0}^{\infty} \frac{s_{2n}}{(2n)!} \{\tanh(k\rho)\}^{2n} \quad , \\ \text{where} \quad \frac{d^{2n}}{dx^{2n}} \left(\frac{x}{\ln \frac{1+x}{1-x}} \right) \Big|_{x=0} &\equiv s_{2n} \quad , \\ s_0 &= \frac{1}{2} \quad , \quad s_2 = -\frac{1}{3} \quad , \quad s_4 = -\frac{12}{5} \quad , \quad \dots \quad , \end{aligned} \quad (41)$$

The above expansion formula is a key equation to solve the REE in the form of (19). All coefficients are finite as shown above. We can take the limit $\rho \rightarrow \infty$ above, and see the infinite series *converges* and gives $0 : \sum_{n=0}^{\infty} \frac{s_{2n}}{(2n)!} = 0$. It is a key, in the present treatment of the infinite series, that we take the expansion using powers of $\tanh \rho$ not those of $e^{-2\rho}$.¹⁶

In the following of this appendix, we take the abbreviated notation $t \equiv \tanh(k\rho)$. Using a relation $t' = \frac{d}{d\rho} \tanh(k\rho) = k(1 - t^2)$, we obtain

$$P' = kv_0(1 - t^2) \sum_{n=1}^{\infty} \frac{e_{2n}}{(2n-1)!} t^{2n-1} \quad ,$$

¹⁶ We can *not* reexpress the RHS of the first equation of (41) as $1/\rho = \sum_{n=0}^{\infty} \frac{s'_{2n}}{(2n)!} e^{-2n\rho}$ with *finite* coefficients. The same notice is said about (19).

$$\begin{aligned}\sigma'' &= k\alpha(1-t^2) \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n)!} t^{2n} = k\alpha \left\{ c_1 + \sum_{n=1}^{\infty} \left(\frac{c_{2n+1}}{(2n)!} - \frac{c_{2n-1}}{(2n-2)!} \right) t^{2n} \right\} \quad , \\ \omega'' &= k\alpha(1-t^2) \sum_{n=0}^{\infty} \frac{d_{2n+1}}{(2n)!} t^{2n} = k\alpha \left\{ d_1 + \sum_{n=1}^{\infty} \left(\frac{d_{2n+1}}{(2n)!} - \frac{d_{2n-1}}{(2n-2)!} \right) t^{2n} \right\} \quad .(42)\end{aligned}$$

Several useful expansion formulae are listed below.

$$\begin{aligned}\frac{\sigma'}{\rho} &= 2k\alpha \sum_{N=0}^{\infty} [sc]_N t^{2N} \quad \text{where} \quad [sc]_N \equiv \sum_{n=0}^N \frac{s_{2n}}{(2n)!} \frac{c_{2N-2n+1}}{(2N-2n+1)!} \quad , \\ [sc]_0 &= s_0 c_1 = \frac{1}{2} c_1 \quad , \quad [sc]_1 = s_0 \frac{c_3}{3!} + \frac{s_2}{2!} c_1 = \frac{c_3}{12} - \frac{c_1}{6} \quad , \quad [sc]_2 = \frac{c_5}{240} - \frac{c_3}{36} - \frac{c_1}{10} \quad , \quad \dots \\ \frac{\omega'}{\rho} &= 2k\alpha \sum_{N=0}^{\infty} [sd]_N t^{2N} \quad \text{where} \quad [sd]_N \equiv \sum_{n=0}^N \frac{s_{2n}}{(2n)!} \frac{d_{2N-2n+1}}{(2N-2n+1)!} \quad , \\ \omega' \sigma' &= \alpha^2 \sum_{N=0}^{\infty} [dc]_N t^{2N+2} \quad \text{where} \quad [dc]_N \equiv \sum_{n=0}^N \frac{d_{2n+1}}{(2n+1)!} \frac{c_{2N-2n+1}}{(2N-2n+1)!} \quad , \\ [dc]_0 &= d_1 c_1 \quad , \quad [dc]_1 = \frac{1}{6} (d_1 c_3 + d_3 c_1) \quad , \quad \dots \\ \omega'^2 &= \alpha^2 \sum_{N=0}^{\infty} [dd]_N t^{2N+2} \quad \text{where} \quad [dd]_N \equiv \sum_{n=0}^N \frac{d_{2n+1}}{(2n+1)!} \frac{d_{2N-2n+1}}{(2N-2n+1)!} \quad , \\ \sigma'^2 &= \alpha^2 \sum_{N=0}^{\infty} [cc]_N t^{2N+2} \quad \text{where} \quad [cc]_N \equiv \sum_{n=0}^N \frac{c_{2n+1}}{(2n+1)!} \frac{c_{2N-2n+1}}{(2N-2n+1)!} \quad , \\ [cc]_0 &= (c_1)^2 \quad , \quad [cc]_1 = \frac{1}{3} c_1 c_3 \quad , \quad \dots \quad , \\ P^2 &= v_0^2 \sum_{N=0}^{\infty} [ee]_N t^{2N} \quad \text{where} \quad [ee]_N \equiv \sum_{n=0}^N \frac{e_{2n}}{(2n)!} \frac{e_{2N-2n}}{(2N-2n)!} \quad , \\ [ee]_0 &= (e_0)^2 \quad , \quad [ee]_1 = e_0 e_2 \quad , \quad [ee]_2 = \frac{1}{12} e_0 e_4 + \frac{1}{4} (e_2)^2 \quad , \quad \dots \quad , \\ P'^2 &= k^2 v_0^2 (1-t^2)^2 \sum_{N=1}^{\infty} [ee]'_N t^{2N} \\ &= k^2 v_0^2 \{ [ee]'_1 t^2 + ([ee]'_2 - 2[ee]'_1) t^4 + \sum_{N=3}^{\infty} ([ee]'_N - 2[ee]'_{N-1} + [ee]'_{N-2}) t^{2N} \} \quad , \\ &\quad \text{where} \quad [ee]'_N \equiv \sum_{n=1}^N \frac{e_{2n}}{(2n-1)!} \frac{e_{2N-2n+2}}{(2N-2n+1)!} \quad ,\end{aligned}$$

$$\begin{aligned}
[ee]_1' &= (e_2)^2 \quad , \quad [ee]_2' = \frac{1}{3}e_2e_4 \quad , \dots \\
(P^2)^2 &= v_0^4 \sum_{N=0}^{\infty} [e^4]_N t^{2N} \quad \text{where} \quad [e^4]_N \equiv \sum_{n=0}^N [ee]_n [ee]_{N-n} \quad , \\
[e^4]_0 &= (e_0)^4 \quad , \quad [e^4]_1 = 2(e_0)^3 e_2 \quad , \quad [e^4]_2 = \frac{1}{6}(e_0)^3 e_4 + \frac{3}{2}(e_0)^2 (e_2)^2 \quad , \dots \quad (43)
\end{aligned}$$

Now we find recursion relations between coefficients by inserting expansion formulae above into the REE (12,13,10).

(A) From (12) we obtain

$$\begin{aligned}
& t^0\text{-part} \quad (c_1, d_1) \\
& k\alpha\{2c_1 + 3d_1\} = 0 \quad , \quad (44) \\
& t^2\text{-part} \quad (c_3, d_3, e_2) \\
& k\alpha\left\{\frac{4}{3}c_3 + \frac{5}{6}d_3\right\} - 2k^2v_0^2(e_2)^2 = k\alpha\left\{\frac{8}{3}c_1 + \frac{5}{3}d_1\right\} - \alpha^2(c_1 - d_1)d_1 \quad , \quad (45) \\
& t^4\text{-part} \quad (c_5, d_5, e_4) \\
& k\alpha\left\{\frac{7}{60}c_5 + \frac{7}{120}d_5\right\} - \frac{2}{3}k^2v_0^2e_2e_4 = k\alpha\left\{\frac{13}{9}c_3 - \frac{1}{5}c_1\right. \\
& \quad \left. + \frac{11}{18}d_3 + \frac{2}{5}d_1\right\} - \alpha^2\left\{\frac{1}{6}(d_1c_3 + d_3c_1) - \frac{1}{3}d_1d_3\right\} - 4k^2v_0^2(e_2)^2 \quad , \quad (46) \\
& t^{2N}(N \geq 3)\text{-part} \quad (c_{2N+1}, d_{2N+1}, e_{2N}) \\
& k\alpha\left\{3\left(\frac{c_{2N+1}}{(2N)!} - \frac{c_{2N-1}}{(2N-2)!}\right) - 2[sc]_N + \left(\frac{d_{2N+1}}{(2N)!} - \frac{d_{2N-1}}{(2N-2)!}\right) + 4[sd]_N\right\} \\
& \quad + \alpha^2\{[dc]_{N-1} - [dd]_{N-1}\} - 2k^2v_0^2\{[ee]_N' - 2[ee]_{N-1}' + [ee]_{N-2}'\} = 0 \quad , \quad (47)
\end{aligned}$$

(B) From (13) we obtain

$$\begin{aligned}
& t^0\text{-part} \quad (c_1, e_0) \\
& 2k\alpha c_1 - \frac{1}{8}\lambda v_0^4(1 - (e_0)^2)^2 = \frac{1}{2}\Lambda \quad , \quad (48) \\
& t^{2N}(N \geq 1)\text{-part} \quad (c_{2N+1}, e_{2N}) \\
& k\alpha\left\{\left(\frac{c_{2N+1}}{(2N)!} - \frac{c_{2N-1}}{(2N-2)!}\right) + 2[sc]_N\right\} - \alpha^2\{[dc]_{N-1} + 4[cc]_{N-1}\} \\
& \quad + \lambda v_0^4\left\{\frac{1}{4}[ee]_N - \frac{1}{8}[e^4]_N\right\} = 0 \quad , \quad (49)
\end{aligned}$$

(C) From (10) we obtain

$$t^0\text{-part} \quad (c_1, e_0)$$

$$4k\alpha c_1 - \frac{1}{4}\lambda v_0^4(1 - e_0^2)^2 = \Lambda \quad , (50)$$

$$\begin{aligned} & t^2\text{-part} \quad (c_3, e_2) \\ 2k\alpha c_3 - k^2 v_0^2 (e_2)^2 + \frac{1}{2}\lambda v_0^4 e_0(1 - e_0^2)e_2 &= 4k\alpha c_1 + 10\alpha^2(c_1)^2 \quad , (51) \end{aligned}$$

$$\begin{aligned} & t^4\text{-part} \quad (c_5, e_4) \\ \frac{1}{6}k\alpha c_5 + \left\{ -\frac{1}{3}k^2 v_0^2 e_2 + \frac{1}{24}\lambda v_0^4 e_0(1 - e_0^2) \right\} e_4 &= \\ 2k\alpha c_3 + \frac{10}{3}\alpha^2 c_1 c_3 - 2k^2 v_0^2 (e_2)^2 - \frac{1}{8}\lambda v_0^4 (1 - 3e_0^2)(e_2)^2 \quad , (52) \end{aligned}$$

$$\begin{aligned} & t^{2N}(N \geq 3)\text{-part} \quad (c_{2N+1}, e_{2N}) \\ 4k\alpha \left(\frac{c_{2N+1}}{(2N)!} - \frac{c_{2N-1}}{(2N-2)!} \right) - 10\alpha^2 [cc]_{N-1} - k^2 v_0^2 \{ [ee]'_N - 2[ee]'_{N-1} + [ee]'_{N-2} \} \\ & + \frac{1}{4}\lambda v_0^4 (2[ee]_N - [e^4]_N) = 0 \quad , (53) \end{aligned}$$

We have now obtained all necessary recursion relations and are now ready for determining all coefficients $(c_{2n+1}, d_{2n+1}, e_{2n})$.

i) t^0 -part, (c_1, d_1, e_0)

Relevant equations are (44),(48) and (50). (48) is, however, equivalent to (50). Hence one coefficient, say e_0 , remains as a free parameter.

$$\begin{aligned} c_1 &= \frac{M^{-4}}{4k\alpha} \left\{ \frac{1}{4}\lambda v_0^4(1 - e_0^2)^2 + \Lambda \right\} \quad , \\ d_1 &= -\frac{2}{3}c_1 \quad , \\ e_0 &= \frac{1}{v_0}P(\rho = 0) : \text{ free parameter} \quad , \end{aligned} \quad (54)$$

ii) t^2 -part, (c_3, d_3, e_2)

Relevant equations are (45),(49) with $N = 1$ and (51). There are two solutions.

Solution 1

$$e_2 = 0 \quad , \quad c_3 = 2c_1 + 5\frac{\alpha}{k}(c_1)^2 \quad , \quad d_3 = -\frac{4}{3}(c_1 + 5\frac{\alpha}{k}(c_1)^2)$$

Solution 2

$$e_2 = -\frac{1}{4}\frac{\lambda v_0^2}{k^2}e_0(1 - e_0^2) \quad , \quad e_0 \neq 0, \pm 1,$$

$$c_3 = \frac{3}{32} \frac{\lambda^2 v_0^6}{k^3 \alpha} M^{-4} e_0^2 (1 - e_0^2)^2 + 2c_1 + 5 \frac{\alpha}{k} (c_1)^2 ,$$

$$d_3 = -\frac{4}{3} (c_1 + 5 \frac{\alpha}{k} (c_1)^2) . \quad (55)$$

Solution 2 includes Solution 1 if we allow $e_0 = 0, \pm 1$.

iii) t^4 -part, (c_5, d_5, e_4)

Relevant equations are (46),(49) with $N = 2$ and (52). They are solved as

$$\begin{aligned} & \{k^2 v_0^2 e_2 + \frac{1}{12} \lambda v_0^4 e_0 (1 - e_0^2)\} e_4 \\ &= k\alpha \left(-\frac{4}{9} c_3 + 2c_1\right) + \alpha^2 \cdot \frac{5}{3} (d_1 c_3 + d_3 c_1 + 2c_1 c_3) \\ & \quad - \lambda v_0^4 \frac{1}{4} (1 - 3e_0^2) e_2^2 + 6k^2 v_0^2 e_2^2 , \\ & \alpha c_5 = -\frac{5}{12} \frac{\lambda v_0^4}{k} e_0 (1 - e_0^2) e_4 + \frac{100}{9} \alpha c_3 + 4\alpha c_1 \\ & + \frac{\alpha^2}{k} \cdot \frac{10}{3} (d_1 c_3 + d_3 c_1 + 8c_1 c_3) - \frac{5}{4} \frac{\lambda v_0^4}{k} (1 - 3e_0^2) e_2^2 , \\ & \alpha d_5 = -2\alpha c_5 + \frac{120}{7} \left\{ \frac{2}{3} k v_0^2 e_2 e_4 + \frac{13}{9} \alpha c_3 \right. \\ & \quad \left. - \frac{1}{5} \alpha c_1 + \frac{11}{18} \alpha d_3 + \frac{2}{5} \alpha d_1 \right. \\ & \quad \left. - \frac{\alpha^2}{k} \cdot \frac{1}{6} (d_1 c_3 + d_3 c_1 - 2d_1 d_3) - 4k v_0^2 e_2^2 \right\} \end{aligned} \quad (56)$$

iv) t^{2N} ($N \geq 3$)-part, $(c_{2N+1}, d_{2N+1}, e_{2N})$

Relevant equations are (47),(49) and (53).

$$\begin{aligned} & k\alpha \left\{ 3 \left(\frac{c_{2N+1}}{(2N)!} - \frac{c_{2N-1}}{(2N-2)!} \right) - 2[sc]_N + \left(\frac{d_{2N+1}}{(2N)!} - \frac{d_{2N-1}}{(2N-2)!} \right) + 4[sd]_N \right\} \\ & + \alpha^2 \{ [dc]_{N-1} - [dd]_{N-1} \} - 2k^2 v_0^2 \{ [ee]'_N - 2[ee]'_{N-1} + [ee]'_{N-2} \} = 0 , \\ & k\alpha \left\{ \left(\frac{c_{2N+1}}{(2N)!} - \frac{c_{2N-1}}{(2N-2)!} \right) + 2[sc]_N \right\} - \alpha^2 \{ [dc]_{N-1} + 4[cc]_{N-1} \} \\ & + \lambda v_0^4 \left\{ \frac{1}{4} [ee]_N - \frac{1}{8} [e^4]_N \right\} = 0 , \\ & 4k\alpha \left(\frac{c_{2N+1}}{(2N)!} - \frac{c_{2N-1}}{(2N-2)!} \right) - 10\alpha^2 [cc]_{N-1} - k^2 v_0^2 \{ [ee]'_N - 2[ee]'_{N-1} + [ee]'_{N-2} \} \\ & + \frac{1}{4} \lambda v_0^4 (2[ee]_N - [e^4]_N) = 0 , \end{aligned} \quad (57)$$

Because $[sc]_N, [sd]_N, [ee]'_N, [ee]_N$ and $[e^4]_N$ have the terms c_{2N+1}, d_{2N+1} and e_{2N} *linearly* (for $N \geq 3$) and other parts of the relevant equations are made of lower-order terms, they are three coupled *linear* equations for the three variables c_{2N+1}, d_{2N+1} and e_{2N} . Hence we conclude that $(c_{2N+1}, d_{2N+1}, e_{2N})$ are fixed by the lower-order coefficients.

11 Appendix B: Numerical Analysis by Runge-Kutta

We explain the numerical analysis of Sec.7 of the text. Instead of the functions σ', ω' and P , it is convenient to use normalized ones:

$$\tilde{P} = \frac{P}{v_0} \quad , \quad \Sigma = \frac{\sigma'}{\alpha} \quad , \quad \Omega = \frac{\omega'}{\alpha} \quad . \quad (58)$$

They satisfy the unit boundary condition: $\tilde{P}, \Sigma, \Omega \rightarrow 1$ as $\rho \rightarrow \infty$. In terms of these functions, we can rewrite the REE equations (12, 13, 10) as

$$\begin{aligned} \frac{1}{k} \frac{d}{d\rho} \Sigma &= -\frac{1}{k\rho} \Sigma + \frac{\alpha}{k} (\Sigma \Omega + 4\Sigma^2) + \frac{1}{8} \frac{\lambda v_0^4 M^{-4}}{k\alpha} (\tilde{P}^2 - 1)^2 + \frac{1}{2} \frac{\Lambda M^{-4}}{k\alpha} \quad , \\ \frac{1}{k} \frac{d}{d\rho} \Omega &= -\frac{4}{k\rho} \Sigma - \frac{2}{k\rho} \Omega + \frac{\alpha}{k} (4\Sigma \Omega + \Omega^2) + \frac{1}{8} \frac{\lambda v_0^4 M^{-4}}{k\alpha} (\tilde{P}^2 - 1)^2 + \frac{1}{2} \frac{\Lambda M^{-4}}{k\alpha} \quad , \\ \frac{1}{k} \frac{d}{d\rho} \tilde{P} &= + \left(-4 \frac{\alpha M^4}{k v_0^2} \frac{1}{k\rho} \Sigma + 4 \frac{\alpha^2 M^4}{k^2 v_0^2} \Sigma \Omega \right. \\ &\quad \left. + 6 \frac{\alpha^2 M^4}{k^2 v_0^2} \Sigma^2 + \frac{1}{4} \frac{\lambda v_0^2}{k^2} (\tilde{P}^2 - 1)^2 + \frac{\Lambda}{k^2 v_0^2} \right)^{1/2} \quad . \quad (59) \end{aligned}$$

This is the standard form of the Runge-Kutta equation for \tilde{P}, Σ and Ω with the coordinate $k\rho$. All dimensionless constants appearing in the RHSs are rewritten, in terms of the dimensionless vacuum parameters in (38): $\sqrt{-\Lambda}/M^2 k \equiv A$, $\lambda v_0^4/M^4 k^2 \equiv L$, $v_0^2/M^4 \equiv B$, as

$$\begin{aligned} \frac{-\Lambda M^{-4}}{k\alpha} &= \sqrt{10} A \quad , \quad \frac{\lambda v_0^4 M^{-4}}{k\alpha} = \sqrt{10} \frac{L}{A} \quad , \quad \frac{\alpha}{k} = \frac{A}{\sqrt{10}} \quad , \\ \frac{\alpha M^4}{k v_0^2} &= \frac{1}{\sqrt{10}} \frac{A}{B} \quad , \quad \frac{\lambda v_0^2}{k^2} = \frac{L}{B} \quad , \quad \frac{-\Lambda}{k^2 v_0^2} = \frac{A^2}{B} \quad , \\ \frac{\alpha^2 M^4}{k^2 v_0^2} &= \frac{1}{10} \frac{A^2}{B} \quad . \quad (60) \end{aligned}$$

The values of (A, B, L) must be properly obtained corresponding to a given boundary condition. For the case $\tilde{P}(k\rho = 0) = e_0 = -0.8$ case (Solution 2), we have obtained their values as $(A, B, L) \sim (\alpha_0, \beta_0, \delta_0) = (1.864786, 0.6424929, 15.90691)$ in Sec.6. We take these values in this appendix. As for the initial condition for $(\tilde{P}, \Sigma, \Omega)$, we analyze two cases.

(i) Ultra-Violet Start

First we numerically solve the differential equation in the direction from the small $k\rho$ value to the large one. Because the origin $k\rho = 0$ is the "superficial" singularity in (59), we must take, as the initial point, some point slightly away (in the positive direction) from the origin. Say, $k\rho = 0.1$. As the boundary value of $\tilde{P}(0.1), \Sigma(0.1)$ and $\Omega(0.1)$, we take the values obtained from the analytic result of Sec.6. They are -0.791057, -0.123938 and 0.082700, respectively. The results are shown in Fig.6,7, and 8 of Sec.7. Those figures surely reproduce the analytical result of Sec.6 up to $k\rho \sim 1$. It strongly supports the correctness of the solution found in Sec.3 and 6. It also means the validity of the truncation approximation to determine the vacuum parameters. The calculation stops with producing a imaginary value. It occurs at the place where the inside of the square root in the last equation of (59) approaches 0 but is still required to be non-negative. The present precision is insufficient to keep non-negative in such nearly 0 region.

(ii) Infra-Red Start

Next we numerically solve the equations in the direction from the large $k\rho$ value to the small one. Say, from $k\rho = 3.1$ to $k\rho = 0.1$. As for the initial values $\tilde{P}(3.1), \Sigma(3.1)$ and $\Omega(3.1)$, one idea is to take the analytic result of Sec.6, as in the (i) case. It, however, fails. The more stringent requirement of the positivity, than the case of the UV start case, is demanded. We can see the situation by writing the inside of the square root of (59) as

$$\begin{aligned} \left(\frac{1}{k} \frac{d}{d\rho} \tilde{P} \right)^2 &= -\frac{4}{\sqrt{10}} \frac{A}{B} \frac{1}{k\rho} \Sigma + \frac{1}{5} \frac{A^2}{B} (3\Sigma + 5\Omega)(\Sigma - \Omega) \\ &\quad - \frac{A^2}{B} (1 - \Omega^2) + \frac{1}{4} \frac{L}{B} (\tilde{P}^2 - 1)^2 \geq 0 \quad . \end{aligned} \quad (61)$$

In the asymptotic region $k\rho \rightarrow \infty$, we know $\tilde{P} \rightarrow 1$, $\Sigma \rightarrow 1$, $\Omega \rightarrow 1$, $\Sigma/k\rho \rightarrow 0$. Therefore the above quantity becomes quite small and, at the same time, is required to be non-negative. The failure means the approximation of the 3-terms truncation used in Sec.6 is insufficient in the IR region. Here we cannot help being content with the qualitative correctness of the solution. In Fig.9,10,11, we show the results for the initial data: the initial point

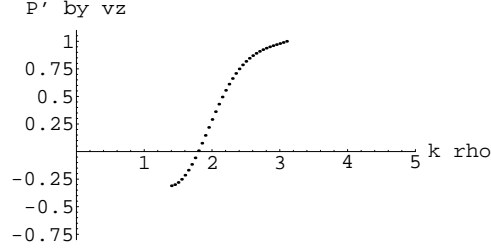


Fig.9 The Higgs Field $P(\rho)/v_0$ obtained by the Runge-Kutta method.
Infra start. Horizontal axis: $k\rho$.

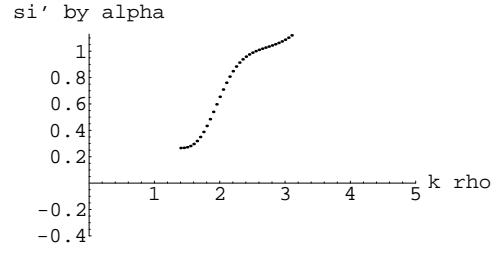


Fig.10 The warp factor $\sigma'(\rho)/\alpha$ obtained by the Runge-Kutta method.
Infra start. Horizontal axis: $k\rho$.

$k\rho = 3.1$; the initial values $\tilde{P}(3.1) = 1.0, \Sigma(3.1) = 1.12, \Omega(3.1) = 1.12$. They are consistent with the analytical result in the qualitative way.

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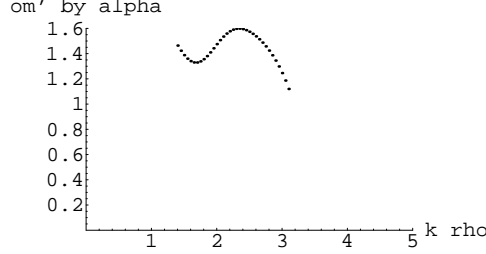


Fig.11 The warp factor $\omega'(\rho)/\alpha$ obtained by the Runge-Kutta method.
Infra start. Horizontal axis: $k\rho$.

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